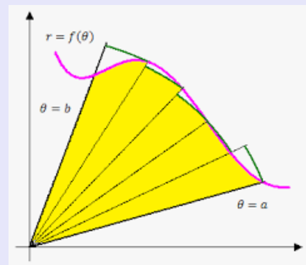
Calculus II

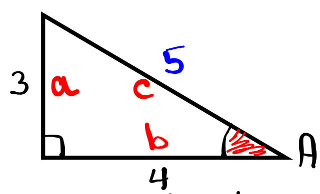
Lecture 4



Feb 19-8:47 AM

Class QZ 4

Given :



Right Triangle

$a^2 + b^2 = c^2$

$3^2 + 4^2 = c^2$

$\boxed{c = 5}$

Find (Ans. in fraction)

1) $\sin A = \frac{\text{Opp.}}{\text{Hyp.}} = \boxed{\frac{3}{5}}$

2) $\cos A = \frac{\text{Adj.}}{\text{Hyp.}} = \boxed{\frac{4}{5}}$

3) $\tan A = \frac{\text{Opp.}}{\text{adj.}} = \boxed{\frac{3}{4}}$

Class QZ 3

Evaluate $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

$u = e^x + 1$ ✓

$du = e^x\,dx$

$= \displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

$= \ln|e^x+1| + C$

Since $e^x+1>0$, No Abs. Value needed.

$= \boxed{\ln(e^x+1) + C}$

Expand $\log_2 \dfrac{32x^2}{\sqrt[3]{yz}}$

Recall

$\log_b \frac{M}{N} = \log_b M - \log_b N$

$\log_b MN = \log_b M + \log_b N$

$\log_b M^p = p\log_b M$

Notice

$32 = 2^5$

$\log_b b = 1$

$= \log_2 32x^2 - \log_2 \sqrt[3]{yz}$

$= \log_2 32 + \log_2 x^2 - \log_2 (yz)^{1/3}$

$= \log_2 2^5 + 2\log_2 x - \frac{1}{3}\log_2(yz)$

$= 5\cdot 1 + 2\log_2 x - \frac{1}{3}\left[\log_2 y + \log_2 z\right]$

$= 5 + 2\log_2 x - \frac{1}{3}\log_2 y - \frac{1}{3}\log_2 z$

Use logarithmic diff. to find $\frac{dy}{dx}$

take ln of both sides, expand, simplify, then take derivative.

$$y = \sqrt{\frac{x-1}{x^4+1}}$$

$^{1/2}$

$$\ln y = \ln \sqrt{\frac{x-1}{x^4+1}}$$

$$\ln y = \frac{1}{2} \ln \frac{x-1}{x^4+1}$$

$$2\ln y = \ln(x-1) - \ln(x^4+1)$$

$$2 \cdot \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x-1} - \frac{4x^3}{x^4+1}$$

$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{x-1} - \frac{4x^3}{x^4+1}\right]$$

$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{1(x^4+1) - 4x^3(x-1)}{(x-1)(x^4+1)}\right] = \frac{y}{2}\left[\frac{x^4+1-4x^4+4x^3}{(x-1)(x^4+1)}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{x-1}{x^4+1}}\left(\frac{-3x^4+4x^3+1}{(x-1)(x^4+1)}\right)$$

Use logarithmic diff. to find $\frac{dy}{dx}$ for

$$y = (\ln x)^{\cos x} \qquad \ln y = \ln(\ln x)^{\cos x}$$

$$\ln y = \overbrace{\cos x \ln(\ln x)}^{\text{Product Rule}}$$

$$\frac{y'}{y} = -\sin x \cdot \ln(\ln x) + \cos x \cdot \frac{\frac{1}{x}}{\ln x}$$

$$y' = y\left[-\sin x \ln(\ln x) + \frac{\cos x}{x \ln x}\right]$$

$$y' = (\ln x)^{\cos x}\left[-\sin x \ln(\ln x) + \frac{\cos x}{x \ln x}\right]$$

Discuss the graph of $y = \ln(1 + x^2)$

Domain $\quad 1 + x^2 > 0 \rightarrow (-\infty, \infty)$

Even function
Sym → Y-axis

$f(x) = \ln(1 + x^2)$

$f(-x) = \ln(1 + (-x)^2) = \ln(1 + x^2) = f(x)$

Y-Int → $x = 0 \rightarrow y = \ln(1 + 0^2) = \ln 1 = 0 \rightarrow (0,0)$

x-Int → $y = 0 \rightarrow \ln(\underbrace{1 + x^2}_{1}) = 0 \quad 1 + x^2 = 1 \quad x^2 = 0$

$x = 0$

$(0,0)$

$y' = \dfrac{2x}{1 + x^2} \qquad y' = 0 \rightarrow 2x = 0 \rightarrow x = 0$

C.P. at $(0,0)$

$$y'' = \frac{2(1 + x^2) - 2x(2x)}{(1 + x^2)^2} = \frac{2 + 2x^2 - 4x^2}{(1 + x^2)^2} = \frac{2 - 2x^2}{(1 + x^2)^2}$$

$$y'' = \frac{2(1 + x)(1 - x)}{(1 + x^2)^2}$$

P.I.P. → $y'' = 0$

$x = 1,\ x = -1$

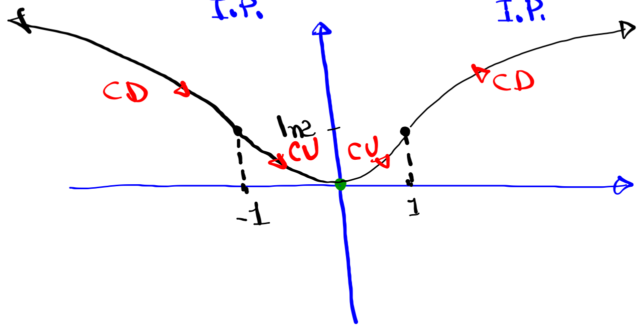
$(1, \ln 2),\ (-1, \ln 2)$

Sign chart

$y' = \dfrac{2x}{1 + x^2} \qquad y'' = \dfrac{2(1 + x)(1 - x)}{(1 + x^2)^2}$

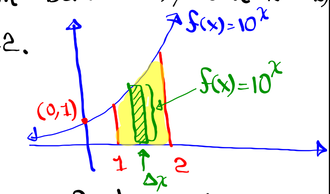
x	$-\infty$ to -1	-1 to 0	0 to 1	1 to ∞
$f'(x)$	−	−	+	+
$f''(x)$	−	+	+	−
$f(x)$				

I.P. $\qquad$ I.P.



Given $f(x) = 10^x$

1) Graph the region below $f(x)$, above x-axis, for $x = 1$ to $x = 2$.



2) Find the exact area of this region.

$$A = \int_1^2 10^x \, dx$$

$$= \frac{1}{\ln 10} \cdot 10^x \Big|_1^2$$

$$= \frac{1}{\ln 10}\left[10^2 - 10^1\right]$$

$$= \boxed{\frac{90}{\ln 10}}$$

$y = a^x$

$\ln y = x \ln a$

$\frac{y'}{y} = \ln a$

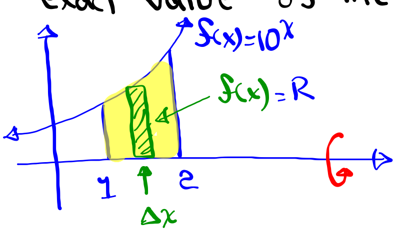
$y' = \ln a \cdot y$

$\frac{dy}{dx} = \ln a \cdot a^x$

$\frac{d}{dx}\left[a^x\right] = \ln a \cdot a^x$

$$\boxed{\int a^x \, dx = \frac{1}{\ln a} a^x + C}$$

3) Rotate that region by x-axis, find exact value of the volume.



Disk

$$V = \int_1^2 \pi \left[10^x\right]^2 dx$$

$$= \pi \int_1^2 10^{2x} \, dx$$

$u = 2x$

$du = 2\,dx$

$\frac{du}{2} = dx$

$$= \pi \int_2^4 10^u \, \frac{du}{2}$$

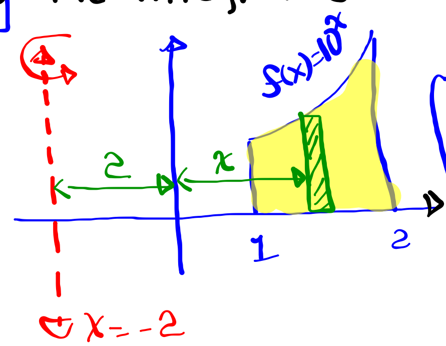
$$= \frac{\pi}{2} \cdot \frac{10^u}{\ln 10} \Big|_2^4 = \frac{\pi}{2\ln 10}\left[10^4 - 10^2\right]$$

$$= \frac{\pi}{\ln 10}(4950)$$

$10^{2x} \neq 10^2 \cdot 10^x$

$10^{2x} \neq 10^{2+x}$

4) Rotate the region about $x=-2$, and $\boxed{\text{Set-up}}$ the integral for the volume.



Shell

$$\boxed{V=\int_1^2 2\pi(x+2)\cdot 10^x\,dx}$$

Find y if $xy^2y' = x+1$

$$xy^2\frac{dy}{dx} = x+1$$

$$xy^2\,dy = (x+1)\,dx$$

$$y^2\,dy = \frac{x+1}{x}\,dx$$

$$\int y^2\,dy = \int \frac{x+1}{x}\,dx$$

$$\frac{y^3}{3} = \int\left(1+\frac{1}{x}\right)dx$$

$$\frac{y^3}{3} = x + \ln|x| + C \quad , \quad x \neq 0$$

$$y^3 = 3(x+\ln|x|+C)$$

$$\boxed{y = \sqrt[3]{3x+\ln|x^3|+C}\,,\quad x\neq 0}$$

If

$f(y)\,y' = g(x)$

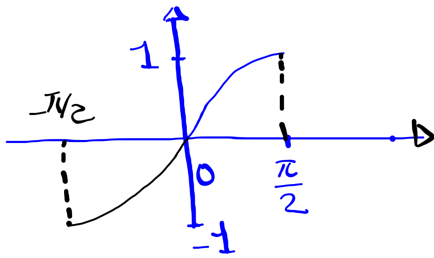
$f(y)\,dy = g(x)\,dx$

take integral.

$\frac{x+1}{x} = \frac{x}{x}+\frac{1}{x}$

$= 1+\frac{1}{x}$

Let's review $y = \sin x$



$y = \sin x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ has an inverse.

$$y = \sin^{-1} x \iff x = \sin y$$

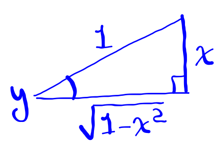
	Domain	Range
$y = \sin x$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[-1, 1]$
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

we like to find

$$\frac{d}{dx}\left[\sin^{-1} x\right].$$

find $\frac{dy}{dx}$ for $y = \sin^{-1} x$ $\rightarrow$ $\frac{x}{1}$ $\frac{\text{opp.}}{\text{Hyp.}}$

$\iff x = \sin y$



Implicit diff.

$$1 = \cos y \cdot \frac{dy}{dx}$$

$$\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\boxed{\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$x \neq \pm 1$

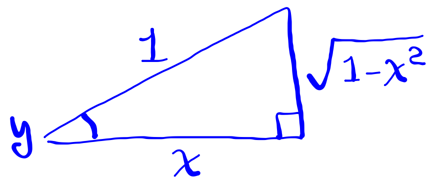
$$\frac{d}{dx}\left[\sin^{-1} e^x\right] = \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x = \frac{e^x}{\sqrt{1-e^{2x}}}$$

$$\frac{d}{dx}\left[\sin^{-1} \ln x\right] = \frac{1}{\sqrt{1-(\ln x)^2}} \cdot \frac{1}{x} = \frac{1}{x\sqrt{1-(\ln x)^2}}$$

find $\frac{d}{dx}\left[\cos^{-1}x\right]$

$y = \cos^{-1}x \to \frac{x}{1}$

$x = \cos y$



$1 = -\sin y \cdot \frac{dy}{dx}$

$\sin y = \sqrt{1-x^2}$

$$\frac{dy}{dx} = \frac{-1}{\sin y}$$

$$\frac{d}{dx}\left[\cos^{-1}x\right] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\cos^{-1}(x^2-1)\right] = \frac{-1}{\sqrt{1-(x^2-1)^2}}\cdot 2x = \frac{-2x}{\sqrt{2x^2-x^4}}$$

$$\frac{d}{dx}\left[\cos^{-1}(\underbrace{\sin x^2}_{u})\right] = \frac{-1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

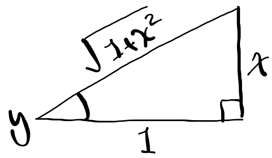
$$= \frac{-1}{\sqrt{1-\sin^2 x^2}}\cdot \cos x^2 \cdot 2x$$

$$= \frac{-1}{\sqrt{\cos^2 x^2}}\cdot \cos x^2 \cdot 2x = \frac{-1}{\cancel{\cos x^2}}\cdot\cancel{\cos x^2}\cdot 2x$$

$$= -2x$$

$$\frac{d}{dx}\left[\tan^{-1}x\right]$$

$$y=\tan^{-1}x \Leftrightarrow x=\tan y \quad \frac{x}{1}$$



$$1=\sec^2 y\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{1}{\sec^2 y}$$

$$\cos y=\frac{1}{\sqrt{1+x^2}}$$

$$\frac{dy}{dx}=\cos^2 y$$

$$\cos^2 y=\frac{1}{1+x^2}$$

$$\frac{dy}{dx}=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left[\tan^{-1}\underbrace{\sqrt{x}}_{u}\right]=\frac{1}{1+(\sqrt{x})^2}\cdot\underbrace{\frac{1}{2\sqrt{x}}}_{\frac{du}{dx}}$$

$$\frac{d}{dx}\left[\tan^{-1}x\right]=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left[\sin^{-1}x\right]=\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\csc^{-1}x\right]=\frac{-1}{x\sqrt{x^2-1}}$$

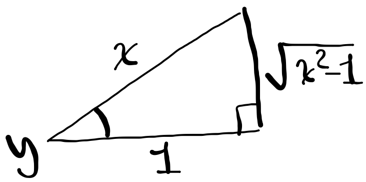
$$\frac{d}{dx}\left[\cos^{-1}x\right]=\frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\sec^{-1}x\right]=\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left[\tan^{-1}x\right]=\frac{1}{1+x^2} \qquad \frac{d}{dx}\left[\cot^{-1}x\right]=\frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left[\sec^{-1}x\right]$$

$$y = \sec^{-1}x \Leftrightarrow x = \sec y$$

$$\cos y = \frac{1}{x}$$



$$\sec y = x$$

$$\tan y = \sqrt{x^2-1}$$

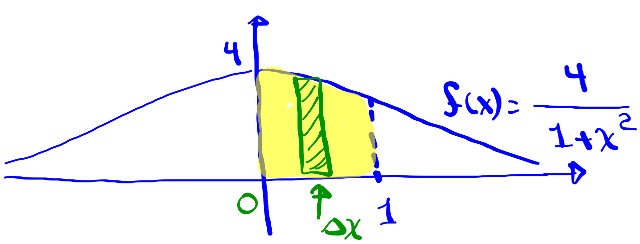
$$1 = \sec y \tan y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left[\sec^{-1}x\right] = \frac{1}{x\sqrt{x^2-1}}$$

Find the exact value of the shaded area below.



$$A = \int_0^1 \frac{4}{1+x^2}\,dx$$

$$= 4\int_0^1 \frac{1}{1+x^2}\,dx$$

$$= 4\tan^{-1}x \Big|_0^1$$

$$= 4\left[\tan^{-1}1 - \tan^{-1}0\right]$$

$$= 4\left[\frac{\pi}{4} - 0\right] = \boxed{\pi}$$

Evaluate $\int \frac{1}{4+x^2}\,dx$

Hint

$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$

$= \int \frac{1}{4\left(1+\frac{x^2}{4}\right)}\,dx = \frac{1}{4}\int \frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx$

$u = \frac{x}{2}$

$du = \frac{1}{2}\,dx$

$2\,du = dx$

$= \frac{1}{\cancel{4}\;2}\int \frac{\cancel{2}}{1+u^2}\,du$

$= \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}u + C$

$= \boxed{\frac{1}{2}\tan^{-1}\frac{x}{2} + C}$

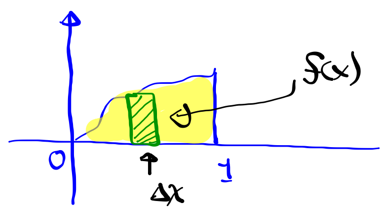
Find the area below $f(x) = \frac{x}{1+x^4}$,

above x-axis, from $x=0$ to $x=1$.

$f(0) = 0$

$f(1) = \frac{1}{2}$

$f(x) > 0 \quad (0,1)$



$A = \int_0^1 \frac{x}{1+x^4}\,dx = \int_0^1 \frac{x}{1+(x^2)^2}\,dx$

$u = x^2$

$du = 2x\,dx$

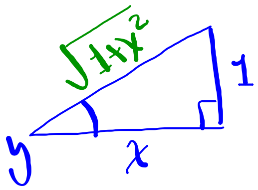
$\frac{du}{2} = x\,dx$

$A = \int_0^1 \frac{1}{1+u^2}\,\frac{du}{2} = \frac{1}{2}\tan^{-1}u\Big|_0^1$

$= \boxed{\frac{\pi}{8}}$

Class QZ 5 (Open Notes)

Show $\frac{d}{dx}\left[\cot^{-1} x\right] = \frac{-1}{1+x^2}$



$y = \cot^{-1} x$

$x = \cot y$

$1 = -\csc^2 y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-1}{\csc^2 y}$

$\frac{dy}{dx} = -\sin^2 y$

$\sin y = \frac{1}{\sqrt{1+x^2}}$

$\sin^2 y = \frac{1}{1+x^2}$

$$\boxed{\frac{d}{dx}\left[\cot^{-1} x\right] = \frac{-1}{1+x^2}}$$